Chapter # 16

PROMOTING MATHEMATICAL MODELLING AS A COMPETENCE: STRATEGIES APPLIED IN PROBLEM SOLVING ACTIVITY

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ABSTRACT
Current international documents claim that schools should enhance mathematical modelling competencies in students, as part of an instructional approach that can be considered suitable for the 21st century learners and problem solvers. The objective of this paper is to identify high-school students' initial conceptions and strategies in mathematical modelling that can be taken into consideration when teachers work out educational activities. A clinical interview approach is applied to understand the modelling strategies that are used by nine students during their three problem solving activities proposed in this research. Students showed different approaches in: their use of algebraic symbolism, the justification of their reasoning, representing their ideas mathematically. The findings can help teachers design school activities that are sensitive to the students' initial conceptions, in order to promote their mathematical modelling competencies.

Keywords: mathematical modelling, initial conceptions, mental processes, clinical interview approach, mathematics education.

1. INTRODUCTION

One of the crucial points in education is to create learning activities in which students are encouraged to apply their declarative knowledge to solve real problems (Vergnaud, 1981; UNESCO, 2012). This process requires the students to recognize the relationship between the real world and the mathematics world, as well as to understand the role that mathematics plays in offering viable and dynamic models of real-world situations (UNESCO, 2012).

The Recommendation of the European Parliament and of the Council (2006) claims to the development of competence in school education. The competence approach is based on a dynamic model of lifelong learning in which new knowledge and skills necessary for successful adaptation to a changing world are continuously acquired throughout life (OECD, 2006). In our view, mathematical competence consists in the understanding of a problematic situation, according to mathematical language. Therefore, competence is not limited to the memorization of mathematical facts and procedures, but it is the flexible capability of applying the relevant knowledge in real world contexts, in order to make inferences, to solve problems and taking sound decisions. We focus on competences because we are especially interested on the learning of mathematics to become not a "academic knowledge", but a way of thinking of the students, even during their daily lives.

According to De Corte (2007), the acquisition of mathematical competence is possible through a classroom environment in which students have the opportunity to learn mathematics as a dynamic discipline rather than the acquisition of a standard system of procedures that are applied to tasks with an expected solving procedure. Modelling can be considered a dimension of the mathematical competence, since it implies the application of mathematical knowledge in real-world situations.

In this research, we analyse initial concepts of modelling strategies developed by students who participated of the present study.

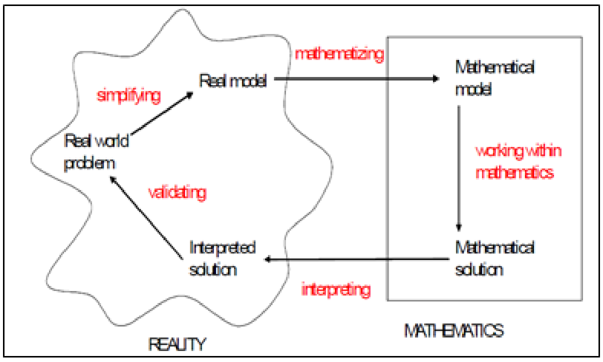
In developing modelling competencies, the students should focus on higher order abilities of translation, interpretation and evaluation of the real life problem in terms of the mathematical model, rather than on algorithmic procedures (Blum, Galbraith, Henn, & Niss, 2007). Furthermore, they must reflect and justify logically the reasoning adopted. Although modelling activities are considered an important educational objective, they are rather rare in mathematics lessons, and there still exists a considerable gap between the ideals of educational debate and the everyday teaching practice (Blum, 2002).

2. THEORICAL BACKGROUND

Analysing the definitions of the mathematical modelling and modelling competence, we can affirm, according to Maaβ (2006), that there is a strong connection between them; the Program for International Students Assessment (OECD-PISA, 2006) perceives modelling as having a symbiotic relationship with mathematical competence. We will soon discuss briefly these two definitions.

Modelling is a procedure that places the real world and mathematics in a constant interaction (Blum, 2002). Mathematically, the process of modelling is defined as a procedure to transform real problems in mathematical problems and solve them, interpreting their solutions in the language of the real world.

Figure 1. Modelling process (Ferri, 2006; Maaβ, 2006).



In mathematics, modelling consists in creating a symbolic representation of a real world situation, which extracts the relevant dimensions of the situation, connects them in order that transformations in the model simulate potential transformations in the reality. Models are hypothetical artifacts that give the solver a partial but useful tool to support reasoning. Through the cognitive actions of “simplifying, structuring and idealizing” the problem at hand, a real model is obtained. By working within mathematics, a mathematical solution can be found, interpreted, and then validated (Ferri, 2006; Maaβ 2006; Blum et al. 2007).

In according to Maaβ (2006), modelling problems are authentic, complex and open problems that relate to reality. By their definition, a model must be considered an approximation of reality, since problematic situations are characterized by manifold factors that can hinder and interfere with the idealized approximation. Generally, the construction of the real model is influenced by one's own mathematical knowledge. Modelling activity mobilizes knowledge and competence of the students in solving an everyday life problem, when they assume an active role on the construction of their own knowledge. Such activity requires a constant reflection on the processes involved to solve the problem and stimulate their ability in planning strategies for more complex solutions (Blum et al., 2007).

Maaβ (2006) defined modelling competencies as "skills and abilities to perform modelling processes appropriately and are goal-oriented as well as the willingness to put these into action" (p.117). The author presented sub-competencies "based on theoretical considerations Blum and Kaiser specify the term modelling competencies by a detailed listing of sub-competencies" (Maaβ, 2006, p. 116) that are described to their understanding of the modelling process:

- Competencies to comprehend real problems and working out a possible model based on reality: construct initial hypothesis in which identify key variables that could potentially influence the problem for then construct relations between them and recognize the relevant information, looking for a simplification of the situation.
- Competencies to set up a mathematical model from the real model: once identified the important variables, mathematize its relevant quantities and their relations, searching for its simplification and reduction where necessary, adopting appropriate mathematical notations.
- Competencies to solve mathematical questions of the mathematical model: to use the mathematical knowledge to adjust the relevant variables and its relations, which is the mathematical model, checking for different forms of the same problem or other ways to solve it.
- Competencies to analyse mathematical results in a real situation: once solved the problem, interpret the numbers within the real contexts, giving real sense to the results; to cast the specific solution into a generalized one; to expose solutions using appropriate mathematical language.
- Competencies to validate the solution: to interact the found solutions into the real work to check the feasibility of the mathematical model, examining it, adjusting it and going through the modelling process whenever the results do not hold; to discuss entire model from scratch.

Blomhøj & Jensen (2007) suggest that competence development as a continuous process. Accordingly, in this research, we are interested in knowing how students have developed their skills in mathematical modelling. The thinking strategies used by students in solving modelling activities are analysed by the theory of conceptual change worked out by DiSessa (1993). In order to explain by which mechanism the students formulate their conceptions, DiSessa (1993) introduces the notion of phenomenological primitive or p-prims. P-prims result from the learner's experience in the world, hence "phenomenological primitives", or "p-prims".

A p-prim is a simple cognitive scheme of qualitative reasoning that helps the student to formulate intuitive or spontaneous conceptions to make sense of a situation (DiSessa, 1988; 1993; Hammer, 1996). In this way, the learner may construct an explanation in response to a single phenomenon, based upon the p-prims that are considered relevant.

According to DiSessa (1993), p-prims are activated when students recognize that sensory data might be interpreted or assimilated within a particular p-prim. In view of existence of primitive ideas, direct experience can play a role in understanding abstract concepts. In our research, the p-prims the students select and apply in mathematical modelling real world situations.

We consider the "p-prims" theory a suitable approach to explain the development of students' competencies, since it can represents the intuitive understanding by a novice who selects only a simplified array of information from a situation, as well as the process of unfolding capability of integrating and selecting aspects of the problem into an expert modelling of the situation. This approach can also make evident the specific obstacles that a student encounters when engaged into real world problems.

3. METHOD

The objective of the present study[1] consists in the identification of constraints and opportunities in working out innovative educational activities that can promote modelling as a competence. Specifically, we sought to identify the major obstacles faced by students in the modelling process, trying to identify the basic elements of knowledge or the p-prims (DiSessa, 1988; 1993) of mathematical modelling in real phenomena. The study also analyses the complex shapes of the modelling competence that students have developed until now.

In order to understand the strategies the students apply in modelling, we developed three mathematical activities. The tasks were constructed in such a way that the same sub-competencies were evaluated. A Clinical Interview approach (DiSessa, 2007) was applied in inquiring students' modelling processes during the activities, allowing each interviewee to reveal his/her natural way in solving the problems.

A Clinical Interview methodology concedes that tasks can be designed that provide the researcher an adequate opportunity to make inferences based on the interviews about students' cognitive processes. According to DiSessa (2007) the Clinical Interview allows for interventions where students were encouraged to elaborate on their statements and judgements. In this way, the methodology provides an opportunity to make valid inferences about students' covert intellectual processes and the gathering data provides for a continual interaction between inference and observation. Therefore, the researcher continually analyses conjectures about the students' thinking and intervenes on any occasion that the problem solving activity of the student cannot be adequately explained by the model presented.

The use of interviewing as a successful tool of research must be accompanied by appropriate learning tasks, serving as genuine challenges for students, allowing them to get involved in the interview. The task situations used in the interview are given below:

[1]This study is part of a Ph.D. research.

Taxi activity[2]*:* the text provides a brief description of a taxi service in function of the cost variation, and presents a table with the tariffs set by the city of Bologna. The problem requires to find the kilometre rate price of a taxi trip and to create a mathematical equation that represents the proposed ride. Students are required to search and analyse information, to apply it on different representations and formulate and justify their assumptions.

Statue activity[3]*:* student are required to use an unconventional unit of measure, in order to estimate the total height of the statue, from an image of a big statue representing a human head and some children playing around it, no additional information provided. Interpreting the situation without the numerical date requires students to search others relevant elements of the problem, to make estimation, to construct new relationships between data, and to make critical assessment of the results.

Travel activity[4]*:* it consists in a written text and a road map. The challenge is to create a journey planning, taking account of the proportions of the distances in the map and justify it. Students are required to select and analyse the relevant information, extracting the unit of the relation cost per kilometer, as a basis to make inferences for different situations.

Nine high school students of different colleges and technical institutes participated in the research. Each student participant was interviewed individually and each interview lasted approximately 60 – 75 min. All interviews were video-recorded and transcribed.

4. RESULTS

For the purpose of the present study, we analysed the students' initial conceptions or p-prims, and the unfolding of their complex shapes of the modelling competence. During the analysis, we consider that the results presented by students should not be tagged as true or false, but instead they should be considered as if the applied modelling was pertinent to the given problem.

According to Smith, DiSessa and Roschelle (1993/1994) the students who exhibited p-prims in their working-out activities are called novice modellers, the others are called expert modellers or expert students.

4.1. Phenomenological primitives in modelling activities

The phenomenological primitives are the basic elements of knowledge on which more complex mathematical competences are constructed. By analysing the nine students' modelling activities, some persistent phenomenological activities can be highlighted:

Working out a possible real model: oversimplification in identifying the variables of the problem; creation of new variables not relevant and inadequate mathematical estimates;

Working out a mathematical model: wrong relationships assumed between variables, use of non-mathematical criteria;

Interpreting mathematical results in a real situation: interpretation attributed to non-relevant factors; unreliable assumptions and generalizations of inconsistent results in reality.

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[2]Activity built by the authors.
[3]Activity adapted from the site http://did.cermat.org/
[4]Activity built by the authors.

Regarding the specific activities, we highlight the following:

Taxi activity: students tend to select factors that are not relevant in the cost of the taxi ride and therefore the related mathematical models are not appropriate.

Figure 2. Fragment resolution present by student MAN in Taxi activity.

I: In your opinion, what are the factors that affect the cost of a taxi ride?
MAN: *[…] the factors that affect the cost of the ride are the luggage you carry on board, e.g. extra heavy packs, maybe some gifts for relatives when traveling, the distance between the origin place and the arrival place, the taxi ride itself because there can be tricky taxi drivers that make a longer ride and the number of people boarding on car. Because if there are more people, the price is lower because the passengers share the cost, while a person spends more money if he is the only traveller.*

The student refers to a particular situation and assumes a naive point of view of the situation: He does not consider the mathematical situation but only his experience in using a taxi. He relies on three p-prims that are not relevant to the situation, that is: the identity of the taxi driver, the number of people aboard and the gifts.

Statue activity: conceptions attributed by students concerns the criteria for mathematical modelling; the selection and estimation of the appropriate unit of measure; its correct application to the object.

Figure 3. Fragment resolution present by student SIM in Statue activity.

SIM: [...] *If it [statue] was large from the head to foot, it would be very large because [the person represented] was one of the most important people in the history of Germany [...] he fought for his ideals.*

I: But how high would it be, in your opinion?

SIM: *Even three / four meters.*

I: How did you figure out that it is approximately 3 or 4 meters high?

SIM: *No, sorry! Let's say a little more, like five or six meters.... Because it would be really important to remind to all people that he was the most important person in Germany [...]*

I: And, how did you guess these 5 or 6 meters? How would you explain it to your partner?

SIM: *Because a building from 5 to 6 meters is very high [...] and this would give the image of what this person represented.*

SIM is struggling with a metacognitive problem: 'what are the mathematic relevance criteria?'. His select historically relevant criteria (that is, Adenauer is an important political leader) that cannot be applied to solve the mathematical problem. When asked about the height of the statue, the subject does not construct a unit of measure from the available information (the approximate heights of the children) and refers only casual justifications, such as: *Five or six meters... Because a building from 5 to 6 meters is very high.*

Travel activity: Many students have inappropriate modelling of the scale relation and the unit of measure of the map, (the scale of the map is written 1 : 35.000.000 and the text gives the definition of "scale”. Therefore, they are not able to construct a proportionally correct model from the map and therefore they make many mistakes.

Figure 4. Fragment resolution present by student MAR in Travel activity.

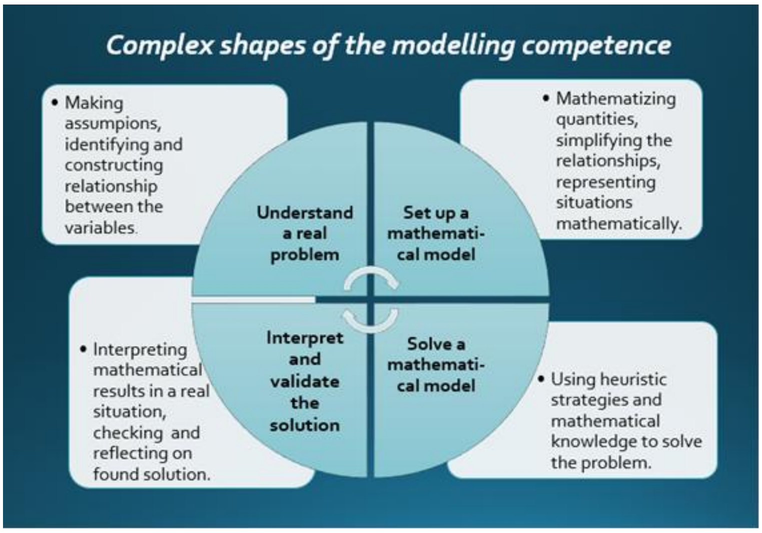
MAR: [...] *I wanted to try…. I do not know … the calculation of how much time you could spend ... I would try to find the time, doing the speed that I invented myself, if you do like 150km / h.* […]
MAR: *So, more or less one week*.
I: How did you come to "more or less one week"?
MAR: *I counted the hours….*
I: How many they are, according to your calculations?
MAR: 8 million.
I: How did you get for the amount "hours" to "one week"?
MAR: *Through the calculator, I counted 24 for 24.... And then… [*I took*] the one closest to the hours. In effect, it would be 24 for 4 but I think it takes more*.

The student tentatively assumes an erroneous average speed, and on this basis she multiplies the speed per hour by the total amount of hours in a week. She multiplies the speed by the total distance, rather than divide it to find the total amount of hours. She assumes a hypothetical average speed, but in her reasoning she considers it as a given and does not revise it in the light of her findings; she puts the numerical data into an incorrect relationship and does not estimate the total amount of hours of driving. As a result she finds a completely unbelievable 'eight million' hours of driving. However, she does reflect on the empirical sense of her findings.

4.2. Complex shapes of the modelling competence

The figure below shows some complex shapes of modelling competence presented by a few students:

Figure 5. Complex shapes of the modelling competence.



An example is the LUC resolution, regarding the Statue activity:

Figure 6. Fragment resolution present by student LUC in Statue activity.

LUC: *If the head of Adenauer is higher by about one and a half children, and the relationship between the head and the body of a person is 1:8 and a child is approximately 1,30 m tall, then:*

$$1:8 = (1{,}30 + 1{,}30 : 2) : x$$

$$1:8 = 1{,}95 : x$$

$$x = \frac{8 \cdot 1{,}95}{1} = 15{,}6 \text{ m}$$

Interpreting the conduct performed in Figure 6: The problem presented a photo and the image is the head of a statue and in front of it there are some children. In order to estimate the statue height, the student constructs the following relationship:

Left side of the equation: He puts in proportion the human body with the human head, saying that the measure of head is 1/8 of the body, and he represents it mathematically as 1:8.

Right side of the equation: He puts in proportion the whole body with the statue head, saying that the statue head is high about a child and half. Since, he considers that the child is high about 1.30 meters, then he writes: 1.30 + 1.30/ 2. He denominates the statue body by x.

He constructs the mathematical model according with the first row of the equation. He solves algebraically the equation and arrives at a valid approximation height.

Expert students present their way of getting to the solution and often clearly exhibit their objectives. They are able to explain the strategic decisions taken and the mental processes used to solve the activities. A limited number of students highlight some possible alternatives that could be more effective to solve the activity.

4.3. General analysis

The interpretation of a real situation remains the main problem for the development of modelling competence. Identifying key variables and constructing a model consistent with the reality are crucial cognitive actions in solving a problem. Some students have selected factors that have no mathematical relation with the problematic situations, such as the solution presented by MAN (Figure 2) regarding the taxi activity. He refers to his own personal experience in taking a taxi: *gifts for relatives, tricky taxi drivers that make a longer ride, the number of people boarding the car to share the cost*. In this case, he refers to a collection of naive abstractions of everyday observed phenomena that do not have a mathematical relevance. Furthermore, he uncritically accepts his experiential p-prims and does not revise his model and does not find the required value: cost per kilometer.

The novice modellers tend to assume mathematical values and do not revise them even in light of absurd results; they are not able to think of numerical values as valid estimations of elements in the real world situations. Making incorrect estimations was the most common mistake made by novices when constructing the mathematical model. When asked if they needed to reconsider the estimations, students relied on their personal experiences to justify their choices. As an example, MAR (Figure 4) made erroneous assumptions throughout her solution procedure and justified it as *I thought about the highway*.

Few students were able to work out a mathematical model from the real situation. As we observed in the Statue problem solved by SIM (Figure 3), the students generally had difficulties in expressing the algebraic equation that represents the real situation. For example, SIM presented a resolution linked to history, without mathematical relevance, stating that *the statue would be really big because it represented one of the most important people in German history*. When the interviewer asked him how high would be the statue, SIM responded *three or four meters*. After having reorganized his response, he introduced a new answer: *5 or 6 meters* and justify it: *Because a 5 to 6 meters building is very high [...] and that would give the right idea of what that person represented.*

In order to create a mathematical model, different competencies of identifying the relevant variables of the problem, of simplifying them if necessary, of choosing an appropriate mathematical notation, of representing the algebraic situation and of formulating and justifying hypothesis are required.

Although all the students have shown arithmetic competence, they express some typical limitations in identifying the vital variables that could possibly describe the real world problem, as the p-prims presented by SIM (Figure 3) and by MAR (Figure 4) make evident. The students were not able to contextualise their intuitions into a mathematical model; they made oversimplification in identifying the variables of the problem, in wrong relationships assumed between variables, in the use of non-mathematical criteria, in interpretations based on irrelevant factors, in inadequate mathematical estimations and in unreliable assumptions and generalizations of inconsistent results in reality.

Expert students successfully interpreted the purpose of the tasks and found a feasible solution, because they related the real world with their mathematical model. A clear example is observed on the Statue problem solved by LUC and reported in Figure 6. Through his explanations, we can understand that he made an appropriate interpretation of the real world and he managed to build the real models in close proximity of reality exhibited. Thereupon, LUC was able to interpret them mathematically, building a model that represents the situation and found valid solutions. It is interesting to look closely the way in which students have mathematized the situation, working out the more efficient one among their own assumptions. It's clear that the “experts” used different kinds of knowledge to correctly apply relevant mathematical principles and definitions. This can be highlighted by comparing SIM’s resolution and LUC’s resolution in the Statue Activity.

We analysed the students' reasoning in real world problems, asking them to construct approximate, but valid models of concrete situations, in order to conduct inferences. The identification of p-prims enables us to understand the critical points for the development of the modelling competence. The clinical interviews with students provided a sensible context to understand the students’ interpretations in terms of p-prims. The exploratory data obtained in the present research do not allow the generalization of the distribution of the p-prims recognized in mathematical modelling, because this inquiry is based on few subjects and is aimed at the detailed description of the phenomenological primitives that are the initial elements of the modelling competence. These mechanisms are simple intuitive abstractions from experience and they are unlikely object of explicit and deliberate considerations by the students.

The study of p-prims can reveal aspects of knowledge not evident through other analytical lenses, yet it demonstrates very practical consequences in atypical mathematics classroom. Engaging students in mathematical discussion has offered us the opportunity to know their p-prims within the modelling process, and therefore we can identify the way they articulate mathematical knowledge during problem solving activities. As the research data have highlighted, it is not enough to have certain skills, in order to solve complex real-world problems, but it's necessary to realize which are the best strategies to be able to get a satisfactory result. One of the obstacles encountered by the students was the difficulty of identifying the relevant mathematical knowledge to construct a viable model of the situation and to conduct inferences. The students participating in the research showed a comparable initial knowledge of mathematical definitions and arithmetical procedures, and therefore potentially can rely upon similar cognitive resources in their modelling activities. However, some students often failed to understand the relevant elements of knowledge to model each problematic situations.

5. FUTURE RESEARCH

Analysing students' modelling competence is a complex field of inquiry and further examinations are still needed in order to compose the range of the initial elements of reasoning in this knowledge domain. The purpose of this exploratory and qualitative study was the recognition of some modelling competences employed by students during the resolution of real-world problems. This study analysed the student individually, but another interesting investigation would be the analysis of students' discussion about a problem, the counterexamples they can make and their explicit justification of relevance for selected elements of knowledge, in order to achieve a feasible solution and to rework their initial p-prims.

Once analysed, the clinical interviews were reported to a group of high school mathematics teachers. The teachers identified some of the obstacles faced by the students in the modelling process and described some instructional methodologies to overcome the obstacles. The group gave us the opportunity to understand how they manage and discover students' modelling competencies. They also explained how the Italian's Curriculum Framework can help them in the classroom. Teachers' suggested that the introduction of modelling competencies can start from the introduction of new arguments by asking students to apply modelling activities. Students can be encouraged to make conjectures in the mathematical reasoning, and can be given more space to expose their reasoning strategies and to make explicit their assumptions. Teachers also recognised that they can reduce the correction of every lacking points of a presented solution, and focusing instead on the students' reasoning strategies. Furthermore, it is suggested to engage students in presenting arguments and counter-arguments in a cooperative process, in order to make their p-prims explicit, object of critical inspection and recontextualisation into a mathematical perspective, by discarding the experiential intuitions that are irrelevant in scientific reasoning.

6. CONCLUSIONS

Modelling is a sophisticated endeavour for students. It involves the understanding of the real world problem, the framing of the appropriate questions, the making of relevant assumptions to simplify the problem, the formulation of models in mathematical language to obtain an adequate solution. The modelling activities provide opportunities for students to apply their knowledge to solve real-world situations. This study has shown that novice students are capable of completing the modelling activity, albeit at different levels of competence.

The emerging data suggest the importance for teachers to be sensitive in recognizing the students' different conceptions and in identifying their mathematical conceptual primitives. From the constructivist point of view, the presence of phenomenological primitives is considered a fundamental aspect of learning and it requires the implementation of effective teaching strategies to promote the development of mathematical competence, by reframing p-prims into a mathematical perspective.

Investigating further the types of knowledge developed by students and the processes they use to apply them are of fundamental importance on designing new teaching practices.

In conclusion, we consider that teaching to promote students' competencies, as it is established by international documents, is a challenging task for all teachers. By adopting a reflective and flexible attitudes, the teachers can identify the role that the experiential intuitions play in students' understanding of the world-problem situations. It is supposed that a constructive environment, in which students can discuss explicitly on the relevance of their p-prims in modelling the problematic situations can enable them to reframe their reasoning in more adequate mathematical perspectives.

In the school context, we are at the beginning of a long process of changing that requires a profound modification of the daily teaching school: a paradigm shift.

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