Chapter # 2

THE IMPACT OF TEACHERS' SUBJECT MATTER KNOWLEDGE ON STUDENTS' LEARNING OF RATIONAL NUMBERS AND PROPORTION

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ABSTRACT
This study examines the impact of teachers' subject matter knowledge on students' learning. The mathematical content deals with rational number as fractions and proportion. The study includes pre- and post-tests from 99 students, classroom observations, students' written solutions and interviews with 48 selected students after the post-tests. Findings from this study show that the impact of teachers' subject matter knowledge and ability to identify the objects of learning, and apply this in teaching, strongly influenced the development of students' conceptual learning about fractions and proportion.

Keywords: teachers' subject matter knowledge, fraction and proportion, students' learning.

1. INTRODUCTION

Empirical studies show the complicity, as well as the importance, of interpreting and transforming subject matter knowledge into teaching. The concept of common content knowledge in instruction, concerning mathematical contextualizing of teaching, and transferring it into more complete specialized content knowledge, has received attention in recent years (Adler & Sfard, 2017; Subramaniam, 2019; Karlsson & Kilborn, 2021). What is emphasized is the importance of teachers' subject matter knowledge and teachers' ability to identify and understand the crucial content in teaching as well as widening and deepening this by appropriate variation. These issues have been discussed earlier in an educational and didactical context by Even (1993) and Lamon (2007) who agreed upon the complicity of teaching a mathematical content with rigor, especially within the area of fraction numbers and proportion. This means that teachers' knowledge of mathematical content with a focus on subject matter is linked to students' learning and gives students opportunities to learn mathematics (Livy & Vale, 2011).

According to Ohlsson (1988), fractions, ratio and proportion are connected areas. However, learning fractions is complicated. To sum up, fractions are difficult to learn and include a long-term process that presupposes learning continuity from grade 1 to grade 9 (Hackenberg & Lee, 2015). As different aspects of fraction and proportion are introduced during different school years, usually by different teachers, there is an urgent need for long-term planning to secure this continuity in teaching and learning. "In general, the researchers found that teachers with a relatively weak conceptual knowledge of mathematics tended to demonstrate a procedure and then give students opportunities to practice it. Not surprisingly, these teachers gave the students little assistance in developing and understanding what they were doing" (Kilpatrick, Swafford, & Findell, 2001, p.377).

Moreover, to understand and operate with rate and proportion, students need a thorough understanding of fractions and how to operate with fractions (Behr, Harel, Post, & Lesh, 1992). This is not an easy task. For students, this demands a long-term process of developing adequate knowledge. For teachers, it requires a thorough knowledge of the mathematics they teach (Ma, 1999), what Shulman (1986) calls subject matter knowledge. Also, other researchers, like Ball, Bass, and Hill, (2004) emphasize the importance of teachers' subject matter knowledge.

Hill, Ball, and Schilling (2008) pointed out that teachers' knowledge of subject matter, related to two components, common content knowledge and specialized content knowledge, is an important prerequisite for students' possibilities to learn. They also emphasized the strong connection between the quality of teaching and teachers' subject matter knowledge, and also that there are different factors in the instructions that entail possibilities as well as challenges for teachers' subject matter knowledge to be used in practice.

2. BACKGROUND LITERATURE VIEW

2.1. Rational Numbers and Proportion

Rational numbers consist of equivalence classes like $\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \frac{18}{24} = \ldots$. An equivalence class can be identified by expanding and/or reducing a given fraction (Ohlsson, 1988). This is an important piece of knowledge to understand proportion.

Direct proportion can be expressed as a relationship between four numbers or quantities in which the ratio of the first pair equals to the ratio of the second pair, written $a{:}b = c{:}d$ or $\frac{a}{b} = \frac{c}{d}$. If $\frac{a}{b}$ equals a constant k, one gets the linear proportion $a = kb$.

Moreover, to solve problems from this field, two conditions are required, a capacity to reason about proportion and find suitable mathematical models, and a capacity to carry out the calculations. Both conditions are related to students' perception of fractions as numbers and as equivalent classes.

In teaching proportion, there is often a focus on part-part, part-whole, whole-whole and the like, during earlier school years, not on important qualities related to the definition of proportion or proportion as a mathematical model. According to Suggate, Davis, and Goulding (2009) this may cause confusion in handling ratio.

The mathematical discourse about fractions and proportions, organized by the teacher, is central for students' learning. This discourse is built on the teacher's common knowledge about fractions, and how to implement it in teaching as specialized knowledge. Specialized knowledge is a bridge between content and students' learning (Depaepe, Verschaffel, & Kelchtermans, 2013; Radovic, Black, Williams, & Salas, 2018).

2.2. Conceptual Change and Conceptual Learning

Learning mathematics is a long-term process, where understanding of fractions is successively experienced by students. Therefore, teaching in grade 8 must rely upon students' pre-knowledge from earlier school years. Referring to Lesh, Post, and Behr (1988), fractions and proportional reasoning are cornerstones of algebra and other areas of mathematics. At the same time fractions are a foundation for proportion. However, as Ohlsson (1988) points out, one difficulty in mastering fractions is "the bewildering array of *many related but only partially overlapping ideas* that surround fractions" (p. 53). This calls for clear and long-term learning of fractions and proportion (Alajmi, 2012; Lee, Choy, & Mizzi, 2021; Berggren, 2022).

When students are taught a new phenomenon, they are often more inclined to assimilate it to their current understanding than to accommodate it with a new and deeper understanding. Posner, Strike, Hewson, and Gertzog (1982) call attention to the process of *Conceptual Change,"* where accommodation may be a process of taking an initial step towards a new conception by accepting, and then gradually modifying, other ideas as they more fully realize the meaning and implication of these new commitments" (p. 223).

A conceptual change is not easy to accomplish, and unsuccessful assimilation could lead to anomalies in students' thoughts. The students conceptual change from fractions to proportion needs conceptual support in teaching (Hiebert & Carpenter, 1992), where teachers' content knowledge in mathematics is a main domain (Grootenboer, 2013).

This study examines the impact of teachers' knowledge of subject matter on students' learning of fractions and proportion from grade 2, 4, 5 to grade 8.

The research questions are:

RQ1. In what ways do teachers transform subject matter knowledge into teaching?

RQ1. What do students experience from teaching about fractions and proportion?

RQ3. What is the relationship between teachers' subject matter knowledge and students' learning from grade 2 to grade 8?

3. THEORETICAL APPROACH AND DESIGN

3.1. Subject Matter Knowledge

Subject matter knowledge includes three different components: c*ommon content knowledge,* s*pecialized content knowledge* and *knowledge at the mathematical horizon.* To choose an object of teaching and find its crucial aspects, teachers need subject matter knowledge (Ball, Thames, & Phelps, 2008). To unpack a content, like fractions as equivalence classes and proportion, and adapt it to students' pre-knowledge, experiences and abilities, teachers need to understand the mathematical concepts and how to express it in teaching. This is also a condition for keeping a focus on "the object of learning" and to offer a suitable variation of the content.

The teaching of fractions and proportion during earlier school years is often based on preliminary, more perceptible, concepts, and such preliminary concepts must gradually be developed into correct mathematical concepts. Hence, it is not enough for teachers to understand the mathematics they are currently teaching, but also to understand it in such a way that the content can be unpacked and developed during later school years (Hill et al., 2008). This makes demands on teachers' ability to overview the development and progression in students' learning from grade 1 and on, to ensure a progression in teaching. It is important to notice, that lack of mathematical content knowledge, like subject matter knowledge, can never be compensated by experiences from practice (Ball, Hill, & Bass, 2005). *Common content knowledge* is a didactical tool for teachers in dealing with a mathematical content and identifying its "objects of learning" and "crucial aspects." *Specialized content knowledge* is a tool for teachers to understand how students learn and how to transform and apply knowledge in teaching praxis.

In this study, particular attention has been paid to teachers' *subject matter knowledge* of fractions and proportion.

3.2. Variation Theory and the Object of Learning

For learning to take place, some crucial aspects of the object of learning need to vary, while others need to remain constant (Marton, 2015). From a teacher's point of view, this requires a good survey of and insight into the actual content as well as opportunities to identify students' multiple conceptions of an actual phenomenon. If not, it is impossible to present a content that enables students to find crucial aspects of the objects of learning or to offer them relevant variation. This means that teachers' subject matter knowledge is crucial for their ability to teach and thus for their students' ability to learn. This is the core of variation theory and its application of teaching and learning in praxis (Marton & Pang, 2006).

Marton (2015) emphasizes that "the object of learning is constituted in the course of learning" (p. 161). To study how an object of learning is understood it is important to relate it to a certain aspect. Inspired by Pong and Morris (2002), the following aspects are chosen: *The expected object* (EO): What are students expected to learn about fractions and proportion? *The intended object* (IO): What do teachers intend to teach according to the chosen object of learning (fraction and proportion)? *The manifest object* (MO): How is the object of learning related to what was really mediated in the classroom? *The experienced object* (XO): What have the students experienced about fractions and proportion from teaching?

3.3. Design of the Educational Materials

To achieve the purpose of this study, an intervention approach was used. The intervention assumed construction of educational materials, EM1 for grade 2, EM2 for grades 4 and 5 and EM3 for grade 8 (all constructed by the researchers). The materials and the purpose of the materials were presented to the teachers during a seminar. Finally, the teachers involved were observed during three lessons each when implementing the EMs.

EM1 deals with an informal introduction of rational numbers and ratios as part-part, part-whole, whole-whole and part of a number (Ohlsson, 1988).

EM2 deals with rational number (fractions) as equivalent classes, extension of fractions, and how to apply ratio and proportion in problem solving (Suggate et al., 2009).

EM3 deals with rational numbers and proportion related to algebraic concepts of ratio and proportionality, and how to use algebraic concepts in problem solving (Ohlsson, 1988).

The aims and goals of EM1, EM2 and EM3 are described in detail in teachers' guides with a focus on crucial aspects of the object of learning, variation of concepts, and how to use this in problem solving. The *expected objects* were given in EM1, EM2 and EM3. The *intended objects* were explained in the teachers' guides and discussed during the seminars.

4. METHODS

4.1. Participants and Data Collection

The study included five classes and five teachers who were rated as highly successful by their principals: T1 (13 students) in grade 2, T2 (21 students) and T3 (20 students) in grade 4, T3 (19 students) in grade 5 and T5 (26 students) in grade 8. The classes were chosen from five schools situated in different suburbs of Stockholm. The reason for this choice of grades was that part-part-whole and part of a number are introduced in grades 1 to 3 before rational numbers and proportion are formally introduced in grades 4 or 5. Lastly, operations with rational numbers and algebraic concepts of ratio and proportionality are more formally taught in grade 8. The study was implemented during five or six successive lessons. Two weeks before that, the teachers were invited to a seminar, where the aims and goals of the study were introduced, and they got access to their EMs. One week before the study was carried out, the students got a pre-test. During the study, all communication between teachers and students was recorded by video, with an extra microphone on the teacher. One week after the study, the students got a post-test and a sample of them was also interviewed. The number of students chosen for the individual interviews varied based on two criteria: (1) their achievement on their post-test, (2) the quality of their answers were especially interesting. The interviews were audio-recorded and transcribed. All collected data were transcribed, categorized, thematized and analyzed according to the methodological design and the theoretical approach.

4.2. Observation schedule

The observation schedule has links to *mathematical content knowledge* (Ball et al., 2008), the theoretical approach with a focus on *subject matter knowledge*, and the methodological design (Marton, 2015). The content of the observations was categorized according to Pong and Morris (2002). The teachers' subject matter knowledge was systematized according to qualitative dimensions showed during the teaching process.

The *common content knowledge* category was interpreted according to: (1) logical structure in the teaching, (2) correct terminology and correct notations, (3) use of correct formulas, and (4) the ability to identify students' conceptions and misconceptions. The *specialized content knowledge* category was interpreted as follows: (1) *identifying* and determining whether *students' non-standard solution* could be generalized; (2) whether students' solution was correct and, if not, in which step the calculation went wrong; (3) identifying patterns in students', wrong solutions and *misunderstandings* and (4) taking measures to give the students *conceptual support* in their teaching.

5. ANALYSIS

The analysis was based on theoretical tools on three levels: (1) To understand the outcome of the teaching process related to teachers' *subject matter knowledge.* (2) To understand students' learning during the lessons and the outcome of pre- and posttest and interviews. (3) To understand relationships between levels (1) and (2). The analysis also includes the relations between teaching as intended and manifest objects and learning as expected and experienced objects. The analysis resulted in answers to research questions RQ1, RQ2 and RQ3.

6. RESULTS

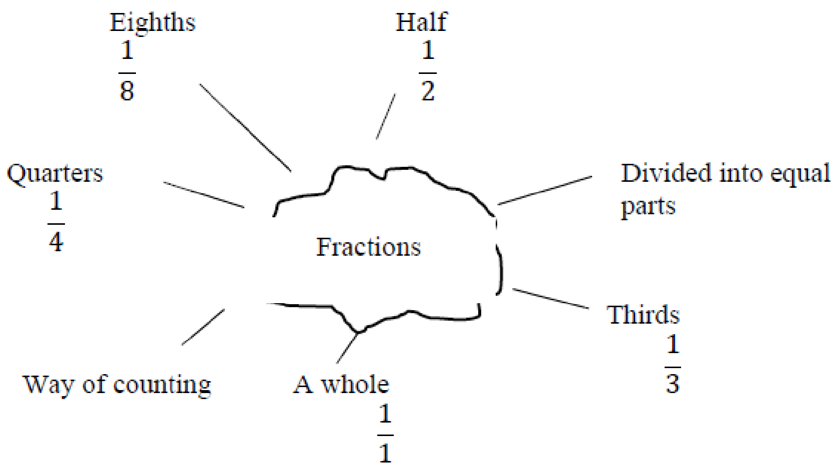
6.1. Teaching and Learning in Grade 2

The mathematical content in EM1 was an informal concept of fraction: part, whole, and part-whole with a focus on part of a number. Before the study, the students worked during one week with fraction as a number and fraction as part-part-whole.

Example 1

At the beginning of the first lesson of the study, teacher and students made a "mind map" as a repetition of what they already knew about fractions. The outcome is presented in figure 1.

Figure 1.
Students' experiences about fractions.



The organization of this discussion had a focus on student's experiences about factions and how to express one or two parts of a whole and how to write this as a fraction. This example showed the students' earlier experiences of fractions (XO).

Example 2

During the first lesson, students had to divide objects in parts. For example, a picture of 12 cinnamon buns in a 3 X 4 pattern was divided into 2, 3 or 4 parts. The students cut the picture into parts which they pasted onto a piece of paper. Later, the students got formal task like *how many are: (a) 2 thirds of 12 buns? (b) 3 quarters of 12 buns?* (IO*).*

The teaching referred to concrete materials and students' early experiences about dividing the cinnamon buns into parts (EO). Characteristics of T1:s teaching is found in table 1.

Table 1.
Characteristics of T1:s teaching.

Teaching and focus on	Mathematical content and concept of fraction	Crucial aspects of fractions and proportion	Common content knowledge	Specialized content knowledge	Indicative communication with students
T1	not enough	enough	not enough	enough	good

The post-tests in grade 2 showed that 9 of 13 students were able to solve tasks like *How many buns are two thirds of 16 buns* and 15 of the students were also able to solve the task, *which is the most, 1 third of 6 apples or 2 sixths of 6 apples?* (XO).

The individual interviews in grade 2 with focus on students' experiences of fractions showed that students felt challenged by tasks without an everyday context because they had no experience of working with two-step tasks, that is, the connection between part-whole and part of number and related how to formally express fractions (XO).

6.2. Teaching and Learning in Grades 4 and 5

EM2 for grades 4 and 5 contains tasks about extending fractions and fractions as equivalence classes. Concerning extending of fractions, the crucial aspect was, that when the denominator is doubled (tripled), the numerator will also be doubled (tripled) like in $\frac{2}{3} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{3 \cdot 2}{3 \cdot 3}$. The aim was that students were expected to continuously explain and discuss this process (IO).

Teacher T2 carried out teaching with a focus on the object of learning and spent almost an hour to ensure that students understood the basics of fractions and how to express fractions before she proceeded. After that, teaching continued fluently without problems. This is confirmed by students' active participation and performance during the lessons.

Here are some examples from teaching in classes T3 and T4.

Example 3

Class T3, grade 4. Discussions of how to express $\frac{1}{3}$.

T3: Which figure shall we write in the numerator?
Students: Six, one, two, three, three, four, six.

Many students were often talking at the same time and were often just guessing (MO).

Example 4

Class T4, grade 5.

T4: How many parts are coloured?
Student: Two.
T4: Out of?
Student: Three.
T4: Two of three (writes $\frac{2}{3}$).

The communication in class T4 was of the type triads, where the students were piloted to fill in a separate number. Consequently, few of them understood connections like $\frac{2}{3} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{3 \cdot 2}{3 \cdot 3}$ (MO).

The teaching in class T2 showed a strong significance regarding the connection between common content knowledge and specialized content knowledge. There were well structured lessons with a stringent focus on the students learning and development. Teaching in class T3 and T4 lacked a clear logical structure and moreover, teachers had difficulty in identifying the students' misconceptions and correct them (MO), see table 2.

Table 2.
Characteristics of T2:s, T3:s and T4:s teaching.

Teaching and focus on	Mathematical content and concept of fraction	Crucial aspects of fractions and proportion	Common content knowledge	Specialized content knowledge	Indicative communication with students
T2	good	good	good	good	good
T3	not enough	not enough	not enough	not enough	not enough
T4	not enough	not enough	not enough	not enough	not enough

The pre-test in classes T2, T3 and T4, showed that some of the students already had some experience of fractions and proportion, while most of them did not. The post-tests showed that T2's students solved and explained most of the tasks correctly (XO), while most of T3's and T4's students had difficulty in finding and explaining their solutions (XO). A noteworthy observation, which was confirmed during the interviews, is that most students in class T3 and T4 really tried to solve the tasks, but most of them had misconceptions of the content (XO), see table 3.

Table 3.
Students' misconceptions in tasks, T2, T3 and T4 classes

Type of misconceptions	$\frac{1}{4}=\frac{2}{8}=\frac{4}{12}$	$\frac{3}{4}=\frac{5}{8}=\frac{7}{12}$	$\frac{3}{4}=\frac{6}{8}=\frac{9}{16}$	$\frac{3}{4}=\frac{4}{8}=\frac{5}{16}$
T2	0	0	0	0
T3	4	2	0	1
T4	6	0	5	2

The individual interviews in grades 4 and 5 showed that most of the students in class T3 and T4 were able to understand and discuss fraction and proportion *when the interviewer supported and guided the students*, for example that (1) a fraction can be written in an infinite number of ways $\frac{1}{4}=\frac{2}{8}=\frac{3}{12}=\ldots$; (2) such an equivalence class will also show the concept of ratio; (3) in this equivalence class a denominator is always four times larger than the numerator (XO).

6.3. Teaching and Learning in Grade 8

The EM3 contained tasks about proportion and proportionality related to problem solving and rational equation. The aim with the tasks was to explain the connection between fractions, proportion and proportionality and apply this in different types of tasks (IO).

Example: "*During a sale the prices were reduced by 10%. Bob paid 63 euro for a pair of shoes. What was the price before the sale?"*

The aim with the task was to introduce and discuss the concept of ratio and proportionality in problem solving and to use different possibilities to carry out the calculation. Instead, the teacher used only the formula $x = \frac{630}{0{,}90}$. It was difficult to identify any focus on proportion or concept of ration and proportionality in the teaching (MO). Moreover, there was limited space for students to communicate or learn different methods during the lessons (XO). The main teaching focus was on formulas and cross-multiplication. However, the meaning of the formulas was not presented to the students, and many of them did not understand when to use them (MO).

It was also clear that students had limited experiences about ratio and proportion. In their textbooks, proportion is defined as $y = kx$. This concept does not work in solving tasks like the one about *Bob's shoes*, above. If proportion had been defined as $\frac{y}{x} = k$, there would have been a simple connection between proportion and ratio, especially if the students had experience of how fractions are classified in equivalence classes. The character of T1's teaching is shown in Table 4.

Table 4.
Characteristics of T5:s teaching

Teaching and focus on	Mathematical content and concepts	Crucial aspects of fractions and proportion	Common content knowledge	Specialized content knowledge	Indicative communication with students
T5	good	not enough	not enough	not enough	not enough

The pre- and post-test in grade 8 showed that most of the students were able to solve simple tasks on ratio and proportion but had difficulty with problem solving that presumed conceptual knowledge about fractions, ratio, proportion and rational equations (XO). The post-tests showed that students just focused on calculation, not on significance or concepts. For example, 11 of 26 students were not able to solve tasks like $\frac{2}{5} = \frac{6}{x}$ or solve a task like "*A flagpole gives a shadow that is 6 meters long. Moa who is 1.50 meter tall gives a shadow that is 1 meter long. How tall is the flagpole*?" However, the most remarkable observation was that none of the students made an outline of the situation with the flagpole. This result confirmed that a basic understanding of concept proportionality is important for students' procedural fluency (XO).

Another observation was that many students in grade 8 had the same experiences and the same types of misconceptions about fractions and proportion that were found among the students in classes T3 and T4 in grades 4 and 5. An interpretation of this is that teachers T3 and T4 were unable to perceive the students' misconceptions (specialized content knowledge). Teacher T5 tried to solve the consequences of such a situation among her students by teaching procedural formulas just fitting to solve predictable problems.

7. CONCLUSION AND DISCUSSION

This study includes intervention with the intention to study content and crucial aspects in teaching and learning fraction and proportion, in the context of conceptual progression and development. Current conceptual contents are: (1) fractions as part, whole, part-whole and part of a number in grade 2; (2) rational numbers as equivalence classes and expanding of rational numbers into proportion in grades 4 and 5; (3) proportionality related to rational equations and problem solving in grade 8. The mathematical context has a focus on teachers' *common content* and *specialized content knowledge* in the teaching. Students' learning in different classroom contexts, related to teachers' subject matter knowledge and teaching style, was central in the observations of the teaching-learning process.

RQ1: In what ways do teachers transform subject matter knowledge into teaching?

One outcome of the study shows that teachers' *common content knowledge* and *specialized content knowledge* were of varying quality. Another outcome is that teacher's *subject matter knowledge* has a decisive influence on how to teach a content and to find the crucial aspects of the object of learning. Teachers T1 and T2 with satisfactory subject matter knowledge were able to organize a classrooms discourse and identify the object of learning and its crucial aspects (Marton, 2015; Ball et al., 2005). On the other hand, teaching in classes T3, T4 and T5 lacked sufficient subject matter knowledge to understand the actual aims and goals. Instead of using the educational materials (EM 2 and EM3) to explain the ideas of equivalence classes (formally or informally) or how to handle ratio in problem solving, they just focused on counting or formulas. Another problem with teaching in classes T3, T4 and T5 was a one-way communication in triads, which made it impossible to reason and to find misconceptions in students' thoughts (Ball et al., 2008; Hill et al., 2008). Even if these teachers tried to follow up the ideas found in the educational materials, they were not able to change their teaching style (Depaepe et al., 2013; Radovic et al., 2018).

RQ2: What do students experience from teaching about fraction and proportion?

The post-test for grade 4 contained tasks of a similar nature as those in the post-test for grade 8. The results of student's achievement in classes T3 and T4, and the results in grade 8 were similar. The errors made by students in grade 8 were often the same as in grades 4 and 5. During the interviews with students in classes T3, T4 and T5 it became evident that many of these errors depended on the same kinds of misconceptions. The most obvious one was students' perception of extending fractions like $\frac{3}{4}$ as $\frac{3+2}{4+2}$. However, the teachers' one-way communication made it difficult to observe these misconceptions and do something about them. At the same time most of students in class T1 and T2 solved most of all tasks. Let us compare this result with the overall result of the post-test in grade 8, where every other student solved almost all problems correctly while most of other students did not solve any. What we learn from this is the importance for students to have suitable pre-requisites (Lesh et al., 1988; Ohlsson, 1988; Grotenboer, 2013) and conceptual continuity in learning about fractions (Alajmi, 2012; Lee et al., 2021; Berggren, 2022). Moreover, when students are introduced to a new concept, they are often more inclined to assimilate it according to their current understanding than develop a new and deeper understanding (Posner et al., 1982; Hiebert & Carpenter, 1992).

RQ3: Which is the relationship between teachers' subject matter knowledge and students' learning from grade 2 to grade 8?

The study showed that subject matter knowledge as phenomena and how it is expressed in teaching had a strong impact on students' learning about fractions and proportion. The relationship between the manifest object (MO) and the experienced object (XO) showed that subject matter knowledge had a positive impact on the intended object and consequently on students' learning (T1 and T2). On the other hand, without sufficient subject matter knowledge or attention to the intended object (T3, T4 and T5), the teaching had a negative impact on students' learning. In order to do something about students' misconceptions, there is a need for a conceptual change, otherwise they may grow and cause anomalies in students' thoughts (Hiebert & Carpenter, 1992; Ball et al., 2005; Hill et al., 2008).

Findings from this study are not general. Researchers consider investigating more about students learning about fractions, especially conceptual shift from fractions to algebra in the teaching context.

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