Chapter #13

ABELIAN GROUPS AND WHAT STUDENT TEACHERS SHOULD LEARN FOR TEACHING ALGEBRA

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ABSTRACT
The purpose of this didactic project is to analyze a current research question, namely how student teachers' knowledge of *Abelian groups* contributes to their understanding of an essential aspect of teaching algebraic concepts by extending numbers and arithmetic operations. The theoretical approaches employed are *Subject Matter Knowledge,* and the choice of algebraic content with focus on students' learning of algebra. Discussions about the *Subject Matter Knowledge* model related to teacher students' learning of algebra in the context of knowledge for teaching are crucial domains for the outcome of this chapter and the research questions. In this chapter a central content is a conceptualization of *Common Content Knowledge* (CCK) related to the algebraic content of the Abelian groups, and its conceptual transformation into *Specialized Content Knowledge* (SCK) for teaching of algebra. Conceptual findings illustrate theoretically the conceptual transformation as interplay between CCK and SCK within the SMK model. This study can contribute with new knowledge about professionally specific mathematic knowledge for teaching algebra. The outcome of this theoretical research work is a follow-up of an earlier research project, namely *Mathematics in teacher education:* S*tudent teachers' knowledge of and perceptions of mathematics*.

Keywords: Abelian group, pre-service teacher, subject matter knowledge, teaching algebra, learning for teaching.

1. INTRODUCTION

The purpose of mathematics teaching is to plan and carry out activities that enable the student to construct and generalize mathematical concepts. For students to successfully learn mathematics, certain crucial aspects and properties of a concept must be varied, while others must remain constant (Davydov, 1990). From a teacher's point of view, this requires a good overview of and insight into the current content, as well as opportunities to identify students' multifaced perceptions of an actual concept. If not, it will be difficult for the teacher to present content that enables the student to find the essence of the concept nor to offer a relevant variation. This is not an easy task for teachers since there is a concurrent ambition to reach a certain level of quality in teaching able to produce long-run effects on students' development of mathematics knowledge (Adler & Sfard, 2016).

This complexity also requires the transformation of formal mathematical concepts to a level that enables students to learn. From a teacher's point of view, this requires a good overview of and insight into the actual content, and an ability to break down the content based on students' individual perceptions of the actual concept. This means that teachers' knowledge of mathematics with focus on content and teachers' subject matter knowledge (Ball & Hill, 2008) include qualitative dimensions. Banner and Cannon (1997) describe, "in order to teach, teachers must know what they teach, and how to teach it; and in order to teach effectively, they must know it deeply and well" (p.7). Other researchers, like Shulman (1986), Ma (2020) and Hill, Rowan, and Ball (2005), have also confirmed the importance of teachers' theoretical and practical competencies for teaching of mathematics. In addition, Ball and Bass (2000) believe that neither the lack of teacher knowledge about mathematical content nor the lack of subject knowledge can be compensated by practical experience. This means that today's teacher education should aim to prepare student teachers in mathematics in such a way that they are able to teach a form of mathematics that favors students' mathematical development.

Mathematics is an abstract and general science for problem solving and method development. The simple addition of $2 + 1 = 3$ is essentially a breathtaking abstraction that applies not only to marbles and apples, but also to hours and days. The abstract nature of mathematics poses a great challenge for teachers to adapt teaching to different students' abilities to think abstractly and absorb the content. To enable this, teacher education should focus on the development of student teachers' own knowledge of mathematics, as well as how this can be transferred into teaching in terms of how different students learn mathematics at different ages (Hill, Ball, & Schilling; 2008).

Mathematics is also part of a cultural heritage and an important tool for perceiving and developing the increasingly complicated world around us. Therefore, a primary aim of teacher education in mathematics is to provide a perspective that is characterized by both mathematics as science and how this can be implemented in school related to students' learning. This means that teacher education should focus both on the development of student teachers' subject-specific knowledge of mathematics and knowledge for teaching in terms of how students learn mathematics at different ages with focus on continuity of learning (Hill et al., 2008). This applies not least to how they can present mathematics in a well-structured way, based on its concrete origins. The goal is for the student teachers to perceive the importance of conceptual sequence in the progression of learning and what students learn during their first years of school, in the long run, gradually will be generalized in the direction of the academic subject of mathematics (Subramaniam, 2019). This implies that the mathematics taught during the first years of schooling must be a preliminary and simplified form of mathematics accessible to all students, but at the same time must be based on sustainable and developable mathematical concepts and methods. Accessibility also deals with student teachers' ability to find continuity in students' learning from pre-school to the 9th grade and onwards. This means that students must successively learn the internal structures of the concepts as essential properties, which in turn, must be generalized with the aim of understanding the significance of the concepts and their connections and relationships to other concepts. To perceive and follow such a learning process in students' learning requires student teachers to be able to process and produce knowledge in their own learning. In other words, they must be able to take a second-order perspective on students' learning, which is intimately related to a first-order perspective on their own learning of mathematics (Leatham, 2006). This is a matter of solid self-awareness of how knowledge is perceived and developed, and the misconceptions that may arise in learning. Students' misconceptions of mathematical concepts and methods can, like incorrect generalizations during earlier school years, cause serious consequences when students reach secondary school. Promoting students' learning of mathematics and perceiving, and correcting their misconceptions requires student teachers not only have solid knowledge of the mathematics they teach, but also of how it could be apprehended by different students in practical teaching.

This complexity also requires a transformation of formal mathematical concepts to a level that enables student teachers to learn. From this point of view, it requires a good overview of and insight into the actual content, and an ability to break down the content, thus, taking student teachers' individual perceptions of the actual concept into account (Askew, 2008). This means that today's teacher education should aim to prepare student teachers in such a way that they are able to teach a form of mathematics that supports the long-term and sustainable development of students' knowledge. Against this background, it is important to know that the student teachers' perceptions and experiences of mathematics often emanate from their own studies of mathematics in primary and secondary school. According to Pajares (1992), such a background often forms a filter through which they apprehend new ideas. This is often crucial for the student teachers' ability to change their understanding of mathematics and assimilate didactics of mathematics in teacher education. The same aspects are problematized by researchers such as Radovic, Black, Williams, and Salas (2018).

2. BACKGROUND

In this chapter, the results of two ongoing theoretical research studies are presented, namely *Teacher students' knowledge of and perceptions of mathematics* (SKUM) and *Mathematics in Teacher Education* (MIL) (Karlsson, 2015). This is followed up with studies on how teachers in different grades teach multiplication, rational numbers, and proportionality, as well as how students in different grades perceive these concepts and how they use them in problem solving (Karlsson & Kilborn, 2018; Klang, Karlsson, Kilborn, Eriksson, & Karlberg, 2021).

These empirical studies identified a need to reconstruct current Teacher Education of mathematics with particular focus on mathematical contexts in order to strengthen student teachers' knowledge about teaching mathematics. Moreover, there is a need to reinforce student teachers' ability to build and analyze mathematical content in their teaching and to carry out sequencing of the content, which can provide development and continuity in students' learning.

The purpose of this theoretical study is to analyze what mathematics in teacher education really means (the SMK model), and how interplay between Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) within the SMK model can be constructed theoretically. There is a certain focus on student teachers' knowledge of mathematical contents, necessary for the sustainable development of students' mathematical knowledge in elementary school. Consequently, there is a particular focus on basic algebra.

Research questions in this theoretical-analytical study are:

(RQ1) What kind of mathematics, specifically algebra, should student teachers learn in relation to the *Subject Matter Knowledge* model?

(RQ2) How can the subject-specific context of Abelian groups (common mathematical knowledge, CCK) provide conceptual support for student teachers to understand essential points for teaching algebra, *Specialized Content Knowledge*, (SCK)?

3. THEORETICAL FRAMEWORK: A PRACTICE-BASED MODEL OF SUBJECT MATTER KNOWLEDGE FOR TEACHING

Over the years, several researchers have claimed that teachers' knowledge of mathematics and knowledge of teaching are not sufficient to develop students' learning of mathematics. This led to a need for a "practice-based" theory called *Subject Matter Knowledge* (SMK) (Shulman, 1986; Ball, Hill, & Bass, 2005). This theory forms a basis for what mathematics teachers should be able to teach. According to Ball, Thames, and Phelps (2008), and Hill et al, (2008), the SMK model is divided into three areas, namely *Common Content Knowledge* (CCK), *Specialized Content Knowledge* (SCK) and *Horizon Content Knowledge* (HCK).

Ball et al. (2008) emphasizes that teachers' CCK is a necessary factor for teaching, but it requires an interaction with SCK. One interpretation of this is that CCK is about the student's perspective of the content, while SCK is about the teacher's perspective of the same content. SCK is a prerequisite for keeping focus on the "learning object" and offering a suitable variation of the content, from lower to higher levels of difficulty. At the same time, it is important to be aware that mathematics taught in the earlier school years is often based on preliminary concepts that will gradually be developed into more correct mathematical concepts. This means that it is not enough for teachers to solely understand the mathematics they teach. They also need to understand it in such a way that the content can be unpacked and developed during later school years. This place demands on teachers' ability to summarize the progression of students' learning from year 1 and onward in order to ensure progression in teaching, which in turn, leads to a need for knowledge in HCK, including knowledge of the curriculum in mathematics. Consequently, HCK is about seeing mathematics from a wider perspective, not least in how the mathematics taught to younger children is connected to teaching at later stages and vice versa. It is also about how basic mathematical patterns (structures) permeate mathematics at all stages.

In this theoretical review, there is a focus on Common Content Knowledge as a prerequisite for understanding Specialized Content Knowledge for teaching mathematics. The analytical transformation of CCK content to SCK is a crucial in this chapter.

3.1. Algebra and Student Teachers Learning for Teaching: Why Algebra?

An important part of teacher training is that student teachers develop algebraic reasoning based on generalized mathematical ideas linked to algebraic concepts. This applies particularly to concepts that constitute the basis of modern algebra and the conceptual relationships between: (1) algebra and the generalization of arithmetic; (2) algebra and patterns; (3) algebra and mathematical models; and (4) the meaning of mathematical symbols (Kaput, 2008). To help students to make such generalizations of arithmetic and understand algebraic content, it is necessary for the student teacher to be provided with sufficient knowledge of algebra to understand relationships between arithmetic and algebra, and how an extension of arithmetic into algebra works in a conceptual sense, before they start to teach algebra (Kieran, 2004). Student teachers' ability to teach algebra depends on their own theoretical conceptual knowledge of algebra. This means that student teachers need to take a teacher's perspective on students' learning to achieve continuity in, and the planned expansion of, algebra in students' learning. This includes conceptual relationships between different number ranges from natural numbers to real numbers, for example, how the basic laws of arithmetic also apply to negative numbers, rational numbers, and real numbers, even if the operations themselves need to be modified. In order to understand negative numbers and rational numbers, it is also important to understand subtraction as the inverse operation of addition, and division as the inverse operation of multiplication. To help students to make such generalizations, it is necessary for the student teachers to be provided with sufficient knowledge of algebra for understanding how an extension of arithmetic in a conceptual sense works, before they start to teach such content (Kieran, 2004). This is also a matter of how students can learn algebra by working informally with natural numbers and rules of arithmetic during younger grades, but in such a way that they will later be able to apply these to whole, rational and real numbers. To understand these generalization processes, the student teachers need meta-knowledge of algebra and an ability to apply this knowledge in the teaching of algebra.

This study draws attention to the interplay between the conceptual sense of algebra (definitions of concepts) as common content knowledge (CCK) and its connection to specialized content knowledge (SCK) in the SMK model. In international research, it becomes evident that in SMK very important factor is given to mathematical structures (mathematical concepts) and the interplay between teacher student conceptual knowledge of fundamental mathematics, and how this knowledge can be used in teaching (Carrillo-Yañez et al., 2018). In Swedish teacher education, the conceptual connection between these two types of knowledge often has a weak character. A reason for this is that mathematics and didactics of mathematics are two different academical disciplines, and the interdisciplinary collaboration between them is insufficient (Bergsten & Grevholm, 2004). In the next chapter, we will illustrate how transformations from fundamental mathematical concepts into teaching can help student teachers to understand mathematics for teaching as well as how this can provide continuity in students' learning of mathematics in a conceptual way. This is a very important aspect of students' learning because their ability to generalize arithmetic into algebra is directly connected to teaching. Moreover, students' individual ability to adapt what is taught relies on teachers' use of correct mathematical structures (concepts). If not, inconsistencies will gradually arise in students' conceptual thinking. A reliable way to avoid this in teacher education is to re-insure the teaching content in basic algebra.

4. KNOWLEDGE FOR TEACHING

4.1. The Abelian Group for Addition

Basic arithmetic assumes two Abelian groups (van der Waerden, 1971), one for addition and one for multiplication (CCK). A group consists of a set, for example, whole numbers, and an operation, such as addition.

The following conditions apply to addition as *Abelian group for addition*.

- For all a and b in the group, the sum $a + b$ also belongs to the group. The group is said to be closed under addition.
- For all a and b in the group, $a + b = b + a$. This is the commutative law.
- For all a, b, and c in the group, $(a + b) + c = a + (b + c)$. This is the associative law.
- There is a neutral element 0 such that $a + 0 = a$ for all a in the group.
- For all a in the group there is an element $(-a)$ in the group such that $a + (-a) = (-a) + a = 0$. Here $(-a)$ is called the (additive) inverse of a and vice versa.

4.1.1. The Abelian Group for Addition and Knowledge for Teaching Addition, its Inverse Subtraction, and Negative Numbers

Understanding of the Abelian group for addition gives student teachers' a key to algebraic ideas and logic. It also gives them an understanding of conceptual algebra and the algebraic content students should learn gradually, which is decisive when introducing algebra as a concept and the essential characteristics of algebra (Karlsson & Kilborn, 2014).

Conceptual finding 1. Addition operation

A closer analysis of the Abelian group for addition shows that the first three points give information about which operations can be performed in addition and how an addition algorithm can be built. This provides important knowledge of what content the teaching should include, with a focus on what students should learn about algebra. Concerning the natural numbers, these three points are known to younger students at an early stage, at least informally.

Conceptual finding 2. Subtraction as the inverse of addition

To progress and understand subtraction as well as how to work with negative numbers, the last two points become important. For every natural number, such as 4, there is an inverse (opposite) number (-4). In this way, negative numbers and subtraction of whole numbers are defined. The subtraction $a - b$ can, for example, be defined as $a + (-b)$. In this way, not only can the negative numbers be defined, but also rules for subtraction of whole numbers. For example, the subtraction $a - b$ can be defined as $a + (-b)$. Based on this, it is easy to explain why $a - (-b) = a + b$. (Notice that b is the inverse of (-b)). Simply stating procedurally that "same signs give plus, and different signs give minus" does not lead to any developable knowledge. Moreover, the two minus signs have completely different meanings, they just look similar.

4.2. The Abelian Group for Multiplication

There is another *Abelian group for multiplication* (van der Waerden, 1971), related to CCK.

- For all a and b in the group, the product $a \cdot b$ also belongs to the group. The group is said to be closed under multiplication.
- For all a and b in the group, $a \cdot b = b \cdot a$. This is the commutative law.
- For all a, b, and c in the group, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is the associative law.
- There is a neutral element 1 such that $a \cdot 1 = a$ for all a in the group.
- For all a in the group (provided that a $\neq$ 0), there is an element $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. $\frac{1}{a}$ is called the (multiplicative) inverse of a and vice versa.

4.2.1. The Abelian Group for Multiplication and Knowledge for Teaching Multiplication, its Inverse Division, and Rational Numbers

Conceptual finding 1. Multiplication operation and the basic law of algebra

As with addition, the first three points of the Abelian group for multiplication are already known by younger students, at least informally, concerning natural numbers. However, they need to learn of the structure and properties of multiplication in the conceptual way.

By studying the definition of the group for multiplication, the student teachers can also realize the risks with a one-sided definition of multiplication as repeated addition.

Multiplication is a two-dimensional operation, whereas addition is just one-dimensional. The commutative and associative laws of multiplication are, for example, difficult to derive from repeated addition because of its one-dimensional nature.

To link addition to multiplication, there is a distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$. An analysis of the definition of multiplication will help the student teachers to understand that multiplication is an arithmetic operation with special properties. The student teachers can also understand the importance of the distributive law, not only to explain how the multiplication algorithm is structured but also its important role in mental arithmetic. For example, $4 \cdot 48 = 4 \cdot (50 - 2) = 200 - 8 = 192$.

Conceptual finding 2. Inverse

The last two points in the definition of Abelian group for multiplication deal with the inverse (reversed) operation of multiplication (Vergnaud, 1983; 1994). With help of the inverse, the student teachers can understand an important property of multiplication, namely by starting from the natural numbers, they can not only define the rule of division of $a \div b$ as $a \cdot \frac{1}{b}$, but also the basic fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ At the same time, the meaning of the multiplication operation must be redefined when the set of numbers is extended from natural numbers to whole numbers, and to rational numbers. To understand this, and thereby create continuity in teaching, it is important that student teachers at all stages study basic algebraic concepts.

Conceptual finding 3. Rational numbers

The inverse operation provide understanding of how algorithms for multiplication and division are structured, as well as basic rules for mental arithmetic. The rules apply not only to natural numbers and whole numbers but also to rational numbers. Notice that for every natural number, such as 4, there is an inverse $\frac{1}{4}$ such as $4 \cdot \frac{1}{4}$, = 1. In this way, not only can fractions be defined, but also division as the inverse of multiplication. By using the inverse, the division $6 \div 4$ can be defined as $6 \cdot \frac{1}{4}$, and the division $\frac{2}{3} \div \frac{4}{5}$ as $\frac{2}{3} \cdot \frac{5}{4}$.

5. FUTURE RESEARCH DIRECTIONS

The findings of the research study draw attention connection between fundamental mathematics and mathematics didactics in the teacher education This is necessary to develop research and implement with focus on student teachers learning for teaching of algebra. This research has opened opportunities for further studies about teaching practice related to learning of subject-specific content, where connections to fundamental mathematics can be used for designing of courses in mathematics didactics for future mathematics teachers. This, in turn, can be the start of a life-long development of teachers' mathematical competences.

This analytical research indicates that student teachers' own knowledge of mathematics and algebra (CCK) is key to their understanding of the content in teaching and what is meant by continuity in and sequencing of the content in mathematics teaching. This means that without an understanding of the conceptual sense of mathematical concepts within algebra (fundamental mathematics), it is impossible to adequately teach algebraic concepts and help students to expand arithmetic into algebra as a path from operations in different number ranges into operations in symbolic form.

6. CONCLUSION/DISCUSSION

An important part of teacher education is a clear definition within the SMK model of what is meant by common content knowledge (CCK) and knowledge for teaching (SCK). It is a key for teacher students to understand the crucial role of their knowledge of mathematics, what is meant by conceptual knowledge, as well as the interplay of conceptual relationships between the concept and different parts of mathematics. This should rely on correct and sustainable algebra like the Abelian groups and how to introduce arithmetic operations within a conceptual algebraic context. This study presents Abelian groups as an important part of modern algebra regarding student teachers' conceptual knowledge of mathematics and how this knowledge can be applied in teaching. There is also an analysis of what mathematics in teacher training can mean in relation to mathematics didactic research, such as the SMK model, and how an interaction between formal mathematics and school mathematics with a focus on algebra can be achieved. Against this background, the student teachers can be offered mathematical content that can be transformed into teaching practice and, in the long run, benefit their future students.

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