Chapter #26

CONCEPTUAL TRANSITION IN STUDENTS' LEARNING FROM ARITHMETIC TO AN ALGEBRAIC CONTEXT
A conceptual way from rational numbers to rational equations

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ABSTRACT
Our current research addresses students' arithmetic and algebraic knowledge, focusing on conceptual connections, and relationships between two aspects of knowledge. The contents in question are rational numbers and rational equations in grades 7, 8 and 9. The study contains three tests given to 400 students in grades 7-9. The tools for analysis comprised an algebraic concept of rational numbers, the theory of generalizing arithmetic into algebra, and theoretical approach about the relationship between arithmetic and algebra in a conceptual context. Current research shows that students' knowledge of algebra and arithmetic has a limited conceptual connection and a weak relationship with each other. Their knowledge of arithmetic operations and solving rational equations used to be solely procedural and relied on formulas learnt in a procedural – and often mixed – manner. This caused conceptual consequences for students' knowledge of rational numbers and their essential properties, as well as shortcomings in students' ability to operate with rational numbers. This study highlights that conceptual transitions from rational numbers to rational equations play a crucial role in students' learning, focusing on the conceptualization of arithmetic concepts and their ability to operate in an algebraic context.

Keywords: rational numbers and algebra, conceptual knowledge, students' arithmetic and algebraic knowledge, conceptual continuity in students' learning.

1. INTRODUCTION

The generalization of algebraic concepts and the ability to create meaning from symbols is a long-term process linked to the expansion of students' arithmetic knowledge (Kieran, 2007). Algebraic reasoning is important for conceptualizing algebra and for using it to expand arithmetic knowledge into abstract algebraic knowledge. According to Mason (2008), the generalization of algebraic patterns calls for concept-based knowledge and the ability to analyze arithmetic situations. This means that students' learning of algebra, related to previous experience of learning and conceptual knowledge, plays a crucial role in operations with rational numbers and solving rational equations (Hackenberg & Lee, 2015). This means, among other things, that students understand conceptual relationships previously used for natural numbers in a way that can be generalized to whole, rational, and real numbers, even if the operations themselves must be modified. At the same time, it is important that students perceive subtraction as the inverse operation of addition, and division as the inverse operation of multiplication. To help students make such generalizations, the teacher must have sufficient knowledge of algebra and understand how an extension of arithmetic works in a conceptual sense, before they start teaching such content (Kieran, 2004). It is also a matter of how students can learn algebra by working informally with the four rules of arithmetic methods in the younger grades, but in such a way that they will later be able to apply this to whole, rational, and real numbers. To understand these generalization processes, students need pre-knowledge about the characteristics of rational numbers before they apply rational numbers in problem solving.

2. BACKGROUND

2.1. Students' Pre-Existing Knowledge

Arithmetics taught during early school years is often based on preliminary and more perceptible concepts, and it is important that these preliminary arithmetic concepts can be gradually developed into correct mathematical concepts. This is often carried out with metaphors or by using different representations, such as pictures. However, according to Kinard and Kozulin (2008), the aim of all representations is abstraction, students' verbal understanding of arithmetical concepts and their crucial properties. Learning rational numbers is a matter of conceptual meaning (Ni & Zhou, 2005; Gözde & Dilek, 2017), a process that successively presupposes adequate pre-existing knowledge of algebra. According to Vygotsky (1986), mathematics is a social construct that implies an ability for abstract thinking. For that reason, students are not able to learn mathematics without support from sufficiently trained teachers.

Students' understanding of rational numbers as arithmetical concepts assumes an ability to think and reason in terms of algebraic abstracts. For students to assimilate the abstract concept of fractions, there is often a need for some kind of representation, a variation of tasks and problem-solving. The aim is to facilitate the verbalization of crucial properties. However, as Ohlsson (1988) emphasizes, fractions are often a "bewildering array", and it is important for a student to know which property of rational numbers is currently represented. For this reason, it is important for students to have suitable pre-existing knowledge of arithmetic (Zazkis & Liljedahl, 2002; Kieran & Martínez-Hernández, 2022). Moreover, when students are introduced to a new phenomenon, they are usually more inclined to assimilate it according to their current understanding than to accommodate and develop a new, deeper understanding (Pajares, 1992).

2.2. Conceptual Continuity in Instruction and Learning

Mathematics is an abstract and general science for problem-solving. This, in turn, is a condition for being general i.e., applicable in a variety of situations. An important follow-up question is what is meant by abstract and abstraction. Skemp (1986) explains the meaning of the terms, linked to school mathematics, as follows, "abstracting is an activity by which we become aware of similarities …among our experiences" and "abstraction is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having similarities of already formed classes" (p. 21).

The case that mathematics is abstract and general does not only apply to the academic subject of mathematics, but also to school mathematics. 2+1=3 is an abstraction that is general in the sense that it is applicable no matter what objects you add, and not only objects, but also minutes, ideas, age, etc. It is important to pay attention to this in students' learning, as well as in formal studies in mathematics. Continuous reflection on relationships between arithmetic and algebra and on the complex nature of an arithmetic problem can be expressed as algebra. This will create conditions for continuity in student learning and provide the knowledge needed for understanding algebra (Carraher, Schliemann, Brizuela, & Earnest, 2006).

A central aspect of mathematics is the field of algebra. A common perception of algebra among students is that it is about complicated "counting with letters". In fact, basic algebra deals with the conditions for the arithmetic operations that students are already learning informally during the first years of school, and how they later can use this to derive and operate with negative numbers and numbers in fractional form. The "letters" are only used to describe the fact that something is general. To describe what is meant by an equation of the first degree does not require a presentation of all such equations. Using symbols, this can be written as $ax + b = 0$, where $a \neq 0$. The conceptual relationship between rational numbers and equations is an important part of students' learning of algebraic symbols and abstracting the ideas behind them (Karlsson & Kilborn, 2014).

Making conceptual generalizations from arithmetic concepts and operations and understanding how an extension of arithmetic works in a conceptual sense, are important parts in students' learning of algebra (Kieran, 2004). They provide conceptual continuity in students' learning of algebra and help them understand symbols and their meaning in equations. It is a matter of how students can learn arithmetic and algebra by working informally with four rules of arithmetic in younger grades, but in such a way that the students later will be able to apply this to whole, rational, and real numbers. To understand this conceptual generalization process in grades 7-9, students need to acquire knowledge about the characteristics of rational numbers before they apply rational numbers in problem solving and in solving rational equations. Therefore, the learning processes such as the transition from arithmetic to algebra are given special attention in the current study.

The purpose of the study is to examine conceptual connections in students' arithmetic and algebraic knowledge of rational numbers, and their ability to use this in problem solving and to solve rational equations. The research questions are: (RQ1) How do students interpret and represent rational numbers? (RQ2) How do students handle transitions from rational numbers to symbols and rational equations? and (RQ3) How do students apply this to problem solving?

3. METHODS

3.1. Participants and Procedure

The study was designed to examine students' arithmetic and algebraic knowledge in a conceptual context with special focus on students' perception of rational numbers and their properties, and how to handle this in solving rational equations and problems dealing with proportion and ratio (Ralston, 2013). The participants were 400 students in grades 7, 8, and 9, with three teachers A, B, and C in 15 classes (see Table 1).

Table 1. Participants.

Class	7a	7b	7c	7d	7e
Teacher	A	A	B	B	C
Number of students in grade 7, n=135					
Class	8a	8b	8c	8d	8e
Teacher	B	B	C	C	A
Number of students in grade 8, n=135					
Class	9a	9b	9c	9d	9e
Teacher	C	C	A	A	B
Number of students in grade 9, n=130					

Table 1 illustrate numbers of students (n) respectively in grade 7, 8 and 9 and their belonging to the teacher A, B or C.

The study includes a quantitative and a qualitative approach. The instrument consists of three diagnostic tests: DT1, DT2 and DT3. Test DT1 focused on representations of rational numbers and operations with rational numbers, test DT2 focused on algebraic equations like $\frac{3}{5} = \frac{x}{8}$, and test DT3 focused on problem solving related to proportion and ratio. Each test consists of 7 tasks of increasing complexity. The tasks were designed with two empty spaces, one for the answer and the other for written explanations of the calculation. The qualitative part consisted of careful analyses of students' answers, the methods they used, and how, why, and when their answers went wrong.

Moreover, the results were interpreted and then explained in the form of a written recommendation intended to develop the skills of the teachers involved as well as their colleagues. The careful and verbatim interpretation of the results on the study together with teachers led to interesting discussions about a practical didactical sense of teaching and how teaching can ensure continuity in students' learning. An important topic in joint discussions related to how teachers can create possibilities for students in grade 7 to repeat and systematize their own pre-existing knowledge and connect this to learning algebra. This question showed that thinking about continuity in teaching is a big challenge for teachers because teachers place a lot of importance on formulas and rote learning methods. This illustrates that the collaboration between researchers and teachers within this project plays a much greater role for the further development of teaching, and joint reflections led to positive effects in ways of thinking about teaching for researchers and teachers alike.

4. THEORETICAL FRAMEWORK

4.1. Generalizing Arithmetic into Algebra

An important feature of teacher training is that student teachers develop skills in algebraic reasoning based on generalizing mathematical ideas linked to algebraic concepts (Blanton & Kaput, 2005). This particularly applies to concepts that constitute the basis of modern algebra, conceptual relationships between algebra, the generalization of arithmetic, algebra and patterns, algebra and mathematical models, and the meaning of algebraic symbols (Kaput, 2008). For students to understand symbols and abstract algebra, they need to generalize algebraic concepts by reasoning with symbols (Kaput, 2008). Students' ability to express themselves using algebra and transform arithmetic concepts into algebraic concepts depends on their conceptual knowledge of the relationships between arithmetic and algebraic concepts, and how numbers are transformed into algebraic symbols. For instance, students' conceptual knowledge of rational numbers is a key to understanding equations, their constructions, and their conceptual meaning. According to Kieran (2004), the generalization of algebra requires algebraic activities with a focus on students' ability to explain and express their knowledge and understanding. Such activities include several main components: (1) generalization of arithmetic concepts; (2) conceptual transformation from arithmetic into algebra; and (3) analyzing and applying this in problem solving. Mastering algebra means not only knowing different algebraic expressions and equations, but also understanding conceptual connections between numbers and expressions and between numbers and equations as tools in problem solving. This means that mastering algebra not only includes a path from separate algebraic expressions and equations to their generalizations, but also the way back – from generalization to arithmetic.

The transformation of student knowledge from arithmetic to algebra presupposes a fundamental understanding of crucial properties and representations of numbers, and their connection to algebraic expressions and equations. Important in Kieran's view of this is that student mastery of algebraic knowledge includes an ability to apply their conceptual knowledge to different problem-solving situations. Such a systematic pattern in students' learning can effectively help them to understand the conceptual relationship between arithmetic and algebra, and how to use this in problem solving. One example is students' conceptual understanding of rational numbers as equivalence classes, such as $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$ etc., which constitutes conceptual pre-existing knowledge in understanding operations such as the extension of rational numbers, conceptual understanding of symbols, and how to add two fractions with different denominators (van der Waerden, 1971). It also offers a method to solve equations like $\frac{x}{2} = \frac{2}{4}$ using algebraic reasoning, and without using procedurally learned formulas (Carpenter & Levi, 2000; Karlsson & Kilborn, 2015).

5. DATA ANALYSIS

The main purpose of the study was to answer research questions RQ1, RQ2 and RQ3 about student conceptual understanding of rational numbers and rational equations, and their ability to use this in problem solving. The theoretical model was based on van der Waerden (1971), Kaput (2008) and Kieran (2004) and was used to analyze and present students' accuracy in tests in a conceptual meaning. The two-level analysis – consisting of quantitative and qualitative parts – enabled researchers to highlight students' conceptual repertoire. The special attention to students' conceptual understanding related to how students perceive the connections between arithmetic and algebra in terms of transition from rational numbers to rational equations and on to problem solving.

6. RESULTS

6.1. Test DT1 (RQ1). Rational numbers and operations with rational numbers

Table 2 shows the correct answers to some chosen tasks.

Table 2. Students' correct answers.

	Task 1 $3 \cdot \frac{2}{7}$	Task 2 $\frac{3}{4} \cdot \frac{2}{7}$	Task 3 $\frac{4}{7} \div 2$	Task 4 $\frac{4}{7} \div \frac{3}{5}$
Grade 7 (n=135)	45%	28%	37%	6%
Grade 8 (n=135)	73%	54%	72%	48%
Grade 9 (n=130)	62%	79%	67%	60%

6.1.1. Test DT1, Qualitative Data

Almost all students in grade 9 relied on formulas to solve the tasks, for example, to solve a simple task such as $3 \cdot \frac{2}{7}$ (Task 1). Moreover, 38% of the students in grade 9 failed to solve that task. A low ability in terms of algebraic reasoning also became evident. In fact, most students tried to use the formula $\frac{a}{b}\ \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ already in grade 7 and most of them failed.

As the formula was learned in a procedural way, it was often mixed up with the formula for division of rational numbers or with cross multiplication.

Just 67% solved the task $\frac{4}{7} \div 2$ in grade 9. Most of them tried to use the formula $\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$. The same method was used already in grade 7. Since this formula was also learned in a procedural way, it was often mixed up with the formula for multiplication of rational numbers or with cross multiplication.

Comments by the authors:
Task 1

A simple conceptual solution to the task $3 \cdot \frac{2}{7}$ is to use repeated addition: $\frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \frac{6}{7}$. This is the same reasoning as 3 · 2 cm = 2 cm + 2 cm + 2 cm = 6 cm.

Task 4

A simple conceptual solution to the task $\frac{4}{7} \div 2$ is $= (\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}) \div 2$ or $(\frac{2}{7} + \frac{2}{7}) \div 2 = \frac{2}{7}$, Like in 6 ÷ 2 = (2 + 2 + 2) ÷ 3 = 2.

6.2. Test DT2 (RQ2). Rational Equations

Table 3 shows the correct answers to some chosen tasks.

Table 3. Students' correct answers.

	Task 1 $\frac{2}{4} = \frac{x}{12}$	Task 2 $\frac{3}{5} = \frac{x}{8}$	Task 3 $\frac{3}{5} = \frac{7}{x}$	Task 4 $\frac{5}{6} = \frac{x}{y}$
Grade 7 (n=135)	63%	9%	7%	2%
Grade 8 (n=135)	47%	8%	1%	7%
Grade 9 (n=130)	83%	40%	7%	3%

6.2.1. Test DT2, Qualitative Data

According to Table 3, 63% of the students in grades 7 and 83% in grade 9 solved the equation $\frac{2}{4} = \frac{x}{12}$ (Task 1). When the denominators were different, it became more difficult. Just 40% of the students in grade 9 solved the equation $\frac{3}{5} = \frac{x}{8}$ (Task 2) and only 7% solved the equation $\frac{3}{5} = \frac{7}{x}$ (Task 3) with x in one of denominators. Most students in grade 9 failed to carry out basic arithmetic operations. Those whose solved the equation $\frac{3}{5} = \frac{x}{8}$ used cross-multiplication. An interesting observation was that few of the same students used cross-multiplication to solve the similar equation $\frac{3}{5} = \frac{7}{x}$ (task 3). Concerning Task 4, students' accuracy was very low in all grades. This kind of task requires students to have a conceptual understanding about what two equal fractions (rational numbers) mean and what proportion means. This confirms a lack of ability for reasoning about and conceptual understanding of the concept of fractions and essential properties of fractions. This also illustrates that solely applying procedurally learned formulas has serious limitations.

6.3. Test DT3 (RQ3). Problem-Solving and Rational Numbers

Table 4 shows the correct answers to some chosen tasks.

Table 4. Students' correct answer.

	Task 1 For 24 kronor you get 3 dl of juice. How much juice do you get for 20 kronor?	Task 2 Anna can cycle 80 km in 3 hours. How long does it take Anna to cycle 50 km at the same speed?
Grade 7 (n=135)	76%	3%
Grade 8 (n=135)	74%	8%
Grade 9 (n=130)	89%	26%

6.3.1. Test DT3 and Qualitative Data

76% of students in grade 7 solved Task 1 through reasoning. Most of them used the constant of proportionality, 8 kronor/dl. However, Task 2 was very difficult to solve for most students. The solution $50 \cdot \frac{3}{80}$ was too complicated like the mathematical model ratio as $\frac{x}{50} = \frac{3}{80}$. This confirms once again that this task requires conceptual understanding about what two equal fractions mean and what proportion means. This kind of conceptual knowledge can also give students understanding about the constant of proportionality.

7. FUTURE RESEARCH DIRECTIONS

This study shows that students' conceptual knowledge in arithmetic and about rational numbers and operations with rational numbers is very important to provide continuity in students' learning (Vygotsky, 1986; Pajares, 1992). It also has an influence on their ability to express arithmetic into algebraic terms in order to understand algebraic equations and use them in problem solving. It also illustrates how students' accuracy in tasks such as those in tests DT2 and DT3 depend on pre-existing knowledge of tasks like the one in test DT1. Conceptualization of arithmetic with rational numbers and its transformation into algebra has been recognized as a crucial yet difficult issue in students' learning of mathematics (Kinard & Kozulin, 2008). This study illustrates that students' pre-existing knowledge of arithmetic (Ohlsson, 1988; Zazkis & Liljedahl, 2002; Kieran & Martínez-Hernández, 2022) and their pre-existing knowledge of rational numbers (Ni & Zhou, 2005; Gözde & Dilek, 2017) play an important role for students in solving algebraic equations and problem solving, and more generally in students' learning of the abstract nature of algebra, expressed in symbols (Carpenter & Levi, 2000; Carraher et al., 2006; Karlsson & Kilborn, 2014; 2015).

An aim in future research is to focus on the didactical aspects of teaching, such as how to plan for better continuity in teaching and how to clarify the relationship between arithmetic and algebraic concepts. The transition in students' learning from rational numbers to proportion and ratio requires careful and proper analysis, like the use of rational numbers in solving rational equations and problem solving connected to ratio and proportion. A careful and proper analysis of this is a crucial conceptual key in students' learning of algebra.

8. DISCUSSION

The analysis of RQ1, RQ2 and RQ3 indicates low conceptual development from grade 7 to 9 in terms of students' ability to handle rational numbers (fractions), rational equations, and algebraic reasoning, as well as in understanding the relationship between rational numbers (fractions) and rational equations, and between fractions and problem solving.

The students' solution of the tasks was largely procedural, essentially only using formulas or methods like cross-multiplication, with few examples of conceptual reasoning. Moreover, a lack of understanding of the formulas and methods resulted in these being mixed up, often causing absurd answers. Students' procedural knowledge also had an influence on the expected progression in their understanding from grade 7 to 9. For example, most students in grade 7 attempted to solve the task $2 \cdot \frac{3}{7}$ by using the formula $\frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7}$ and often made mistakes, like 38% of the students in grade 9.

A low ability in terms of algebraic reasoning also became clear in problem solving. Most students simply tried to apply a formula that they did not know how to use. For example, only 26% of the students in grade 9 were able to solve the task "*Anna can cycle 80 kilometers in 3 hours. How long does it take Anna to cycle 50 kilometers at the same speed*?". The descriptions of their solutions show that most of the students were unable to reason, choose a correct formula, or perform the correct calculation. When comparing the solutions in grade 7 and 9, it became obvious that there had been very little development of knowledge from grade 7 to 9. In grade 7, the students already used the same formulas as in grade 9. The problem with such procedural knowledge is that it offers insufficient grounds for developing algebraic reasoning and make abstracting (generalization) of rational numbers and rational equations (Skemp, 1986; Kieran, 2004).

One crucial task was "*For what values of x and y are* $\frac{5}{6} = \frac{x}{y}$". The response rate was low in all grades. All students who solved the task answered x = 10, and y = 12. This confirms a lack of both reasoning ability and conceptual understanding of fractions (Kaput, 2008). It also shows the students' limited understanding of the important property of fractions as equivalence classes, which is a gateway to understanding and solving the current rational equations.

9. CONCLUSION

The purpose of the study was to examine conceptual connections of students' arithmetic and algebraic knowledge of rational numbers, and their ability to use this in problem solving and to solve rational equations. The results show that the transition from arithmetic to algebra is a difficult process for students and impossible to carry through with only procedural knowledge. Van der Waerden (1971), Kieran (2004) and Kaput's (2008) theoretical frameworks visualize fundamental conceptual limitations in students' solutions of rational equations and their dependence of conceptual knowledge. More specifically, the generalization of algebra cannot take place without the generalization of arithmetical concepts (rational numbers). However, conceptual knowledge of rational numbers implies students' ability to achieve solutions for equations through reflection and reasoning, even without use of formulas.

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